

Random Kneser graphs and hypergraphs

Andrey Kupavskii*

Abstract

A Kneser graph $KG_{n,k}$ is a graph whose vertices are all k -element subsets of $[n]$, with two vertices connected if and only if the corresponding sets do not intersect. A famous result due to Lovász states that the chromatic number of a Kneser graph $KG_{n,k}$ is equal to $n - 2k + 2$. In this paper we discuss the chromatic number of random Kneser graphs and hypergraphs. It was studied in two recent papers, one due to Kupavskii, who proposed the problem and studied it in the graph case, and the more recent one due to Alishahi and Hajiabolhassan. The authors of the latter paper had extended the result of Kupavskii to the case of general Kneser hypergraphs. Moreover, they have improved the bounds of Kupavskii in the graph case for many values of parameters.

In the present paper we present a purely combinatorial approach to the problem based on blow-ups of graphs, which gives much better bounds on the chromatic number of random Kneser and Schrijver graphs and Kneser hypergraphs. The central idea of using blow-ups is due to Noga Alon.

1 Introduction

Kneser graphs and hypergraphs are very popular and well-studied objects in combinatorics. Fix some integers n, k, r , $r \geq 2$. The set of vertices of a Kneser r -graph $KG_{n,k}^r$ is the set of all k -element subsets of $[n]$, denoted by $\binom{[n]}{k}$. The set of edges of $KG_{n,k}^r$ consists of all r -tuples of pairwise disjoint subsets. Thus, for the hypergraph to be non-empty we should assume that $n \geq kr$. Substituting $k = 1$ in the definition gives a complete r -uniform hypergraph on n vertices. When talking about the graph case, that is, $r = 2$, we omit the superscript in the notation of Kneser graphs. For a hypergraph H we denote by $\chi(H)$ its *chromatic number*, that is, the minimal number of colors needed to color the vertices of H without making any edge of H monochromatic.

*Laboratory of Advanced Combinatorics and Network Applications at Moscow Institute of Physics and Technology, Ecole Polytechnique Fédérale de Lausanne; Email: kupavskii@yandex.ru Research supported by the grant RNF 16-11-10014.

The studies of this area started from Kneser graphs. They earned their name from Martin Kneser, who investigated them in the paper [18]. He showed that $\chi(KG_{n,k}) \leq n - 2k + 2$ and conjectured that this bound is tight. This conjecture (or rather its resolution) played a very important role in combinatorics. It was confirmed by L. Lovász [22], who, in order to resolve it, introduced tools from algebraic topology to combinatorics.

After the result of [22] there was a burst of activity around Kneser graphs. I. Bárány [5] gave an elegant proof of Lovász' result, and several authors studied the chromatic number of Kneser graphs of arbitrary set systems (that replace the family $\binom{[n]}{k}$ from the definition). In particular, there were results by V. Dol'nikov [11] and A. Schrijver [26].

In his paper, Schrijver studied Kneser graphs $SG_{n,k}$ constructed on the family of all k -element *stable sets* of a cycle C_n . In other words, the Schrijver family contains all k -element sets that do not have two cyclically consecutive elements of $[n]$. Schrijver noted that a slight modification of Bárány's proof allows for a stronger statement: $\chi(SG_{n,k}) = n - 2k + 2$. In the harder part of the paper, he also showed that $SG_{n,k}$ is a vertex-critical subgraph of $KG_{n,k}$, which means that any proper induced subgraph of $SG_{n,k}$ has strictly smaller chromatic number.

The coloring of $KG_{n,k}$ in $n - 2k + 2$ colors is easy to obtain: for each $1 \leq i \leq n - 2k + 1$ color the remaining sets containing i in color i , and the sets that form the family $\binom{[n-2k+2, n]}{k}$ color in the color 0. A similar coloring for $KG_{n,k}^r$ gives an upper bound $\chi(KG_{n,k}^r) \leq \lceil \frac{n-r(k-1)}{r-1} \rceil$: for $1 \leq i \leq n - kr + 1$ color the remaining sets containing one of the elements $(r-1)(j-1) + 1, \dots, (r-1)j$ in the color j , and the sets from $\binom{[n-rk+2, n]}{k}$ color in the color 0. P. Erdős [13] conjectured that this bound is sharp for all $r \geq 2$. After some partial progress it was confirmed in full generality by N. Alon, P. Frankl, and L. Lovász [4]. The proof again used topological tools.

Generalizing both the result of Dolnikov [11] and Alon, Frankl, and Lovasz [4], I. Kříž [19], [20] obtained the bound on the chromatic number of Kneser hypergraphs of general set families. Later, an elegant proof was obtained by J. Matoušek [24], and some very general results with combinatorial proofs were obtained by G. Ziegler [27]. We also refer to an amazing book written by J. Matoušek on the subject [23].

In [2] N. Alon, L. Drewnowski and T. Łuczak have applied results on colorings of Kneser-type hypergraphs for constructing certain ideals in \mathbb{N} . The hypergraphs they considered are called *s-stable Kneser hypergraphs*, and they may be seen as a generalization of Schrijver graphs. The set family that defines an *s-stable Kneser r -hypergraph* $KG_{n,k}^{r, s\text{-stable}}$ consists of all k -element subsets $\{i_1, \dots, i_k\} \subset [n]$, which consecutive elements are sufficiently far apart: if $1 \leq i_1 < \dots < i_k \leq n$, then for any $j = 0, \dots, k-1$ the elements satisfy $i_{j+1} - i_j \geq s$, as well as $i_1 + n - i_k \geq s$. The case $r = s = 2$ corresponds to the case of Schrijver graphs.

In their paper Alon, Drewnowski and Łuczak proved and applied the following result: $\chi(KG_{n,k}^{r, r\text{-stable}}) = \chi(KG_{n,k}^r)$ for $r = 2^t$, $t \in \mathbb{N}$. They have also stated explicitly the conjecture tracing back to Ziegler's paper [27], which says that the same equality holds for

any r . We are going to use this result of [2], and we note that it would have improved some of the bounds in this paper, if the conjecture was verified. A more general conjecture was made by F. Meunier [25]: $\chi(KG_{n,k}^{r, s\text{-stable}}) = \lceil \frac{n-s(k-1)}{r-1} \rceil$ for any $s \geq r$. It is proved in some cases, but is still wide open in general.

In fact, Kneser was not the first to ask a question concerning Kneser graphs. P. Erdős, C. Ko, and R. Rado [14] proved that the size of the largest family of k -element subsets of $[n]$ with no two disjoint sets is at most $\binom{n-1}{k-1}$, provided that $n \geq 2k$. In terms of $KG_{n,k}$, they determined the independence number of this graph, that is, the maximal size of a subset of vertices not containing an edge of the graph. Later, Erdős [12] asked a more general question: what is the size of the largest family of k -element subsets of $[n]$ with no r pairwise disjoint sets? This is obviously a question about the independence number of $KG_{n,k}^r$, and, unlike the question on the chromatic number, it does not have a complete solution yet. However, the question was resolved for a wide range of parameters by P. Frankl [15]. For some recent progress on the subject see [16], [17].

An r -uniform Kneser hypergraph of any k -uniform set system is an induced subgraph of $KG_{n,k}^r$, and thus the results on the chromatic number of induced subgraphs of $KG_{n,k}^r$ belong to the class of results discussed above. But what if we delete edges, and not vertices? The most natural model to study is that of a *random hypergraph*. For an abstract hypergraph H and a real number p , $0 < p < 1$, define the random hypergraph $H(p)$ as follows: the set of vertices of $H(p)$ coincides with that of H , and the set of edges of $H(p)$ is a subset of that of H , with each edge from H taken with probability p . The results on random graphs and hypergraphs, roughly speaking, tell us how does a *typical* subgraph of a given (hyper)graph that contains a p -fraction of edges behave with respect to a given property. The theory of random graphs and hypergraphs is very rich in both results and open problems, and by no means we are going to give an overview of the field in this note. We refer the reader to the books [3], [6] for some classical results on the subject.

One class of questions that is particularly relevant for this paper deals with transference results. In general we speak of transference if a certain combinatorial result holds with no changes in the random setting. One example of such theorem is due to Bollobás, Narayanan and Raigorodskii [10]. They studied the size of maximal independent sets in $KG_{n,k}(p)$, and showed that for a wide range of parameters the independence number of $KG_{n,k}(p)$ is *exactly* the same as that of $KG_{n,k}$, given by the Erdős–Ko–Rado theorem. Later on, their result was further strengthened by Balogh, Bollobás, and Narayanan [7].

In the paper [21], A. Kupavskii studied the behaviour of the chromatic number of $KG_{n,k}(p)$ and $SG_{n,k}(p)$, showing that, compared to $\chi(KG_{n,k})$, it does change at most by a small additive term in a very wide range of parameters.

Random subgraphs of more general graphs $K(n, k, l)$ were investigated by Bogolyubskiy, Gusev, Pyaderkin and Raigorodskii in [8, 9]. The vertices of $K(n, k, l)$ are k -element

subsets of $[n]$, with two vertices adjacent if the corresponding sets intersect in exactly l elements. In [8, 9] the authors obtained several results concerning the independence number and the chromatic number of $K(n, k, l)(p)$.

In a recent paper [1], which motivated in part the present paper, M. Alishahi and H. Hajiabolhassan generalized the results of [21] to the case of Kneser hypergraphs of arbitrary set systems. They have also strengthened the results of [21] for the graph case. The proofs of Alishahi and Hajiabolhassan are quite technically involved and not easy to follow.

In this paper we describe a purely combinatorial approach to the problem, which allows us to improve significantly the previously known bounds on the chromatic numbers of random graphs in the most interesting cases: for (complete) Kneser and Schrijver graphs and Kneser hypergraphs. Our method may be extended to more general classes of Kneser hypergraphs, which we discuss in Section 6. This does not, however, cover all generalized Kneser hypergraphs, so the result of Alishahi and Hajabolhassan remains best known for some cases.

2 The old and the new bounds

In this section we discuss both the old and the new numerical bounds on the chromatic number of Kneser and Schrijver graphs and hypergraphs. We do not state the bounds in full generality, as they depend on too many parameters and thus are very difficult to interpret. We preferred clarity to generality, and instead focused on several most interesting (in our taste) cases. These cases were also discussed in [21] and [1], so we can compare the new and the old results. The bounds in their full generality appear in the latter sections.

For the rest of the section we assume that $r \geq 2, p > 0$ are fixed. Note that in the case $r = 2$ we formulate our results for Schrijver graphs $SG_{n,k}(p)$. The same bounds hold for Kneser graphs, since Schrijver graphs are subgraphs of Kneser graphs.

We henceforth use the notation $f(n) \gg g(n)$ in a slightly unconventional way. This inequality should be read as: there is a sufficiently large constant C , independent of n but depending on the context of the inequality, such that $f(n) \geq Cg(n)$ for all sufficiently large n . All logarithms with a base that is not specified have the base e . We also do omit writing a.a.s. (asymptotically almost surely) all the time for the statements concerning the chromatic number of hypergraphs.

Returning to the results on colorings, Kupavskii [21] proved that a.a.s.

$$l = 1: \quad \chi(SG_{n,k})(p) \geq \chi(KG_{n,k+1}) = \chi(KG_{n,k}) - 2 \quad \text{if } n - 2k \ll \sqrt{n}; \quad (1)$$

$$\text{fixed } l: \quad \chi(SG_{n,k})(p) \geq \chi(KG_{n,k+l}) = \chi(KG_{n,k}) - 2l \quad \text{if } k \gg n^{\frac{3}{2l}}, \quad l \text{ is fixed}; \quad (2)$$

$$\text{fixed } k: \quad \chi(SG_{n,k})(p) \geq \chi(KG_{n,k+l}) = \chi(KG_{n,k}) - 2l \quad \text{if } l \gg n^{\frac{3}{2k}}, \quad k \text{ is fixed}. \quad (3)$$

Note that in the last two k and l are simply interchanged in the conditions needed for the inequality to hold. The following bounds were proven by Alishahi and Hajiabolhassan [1]:

$$l = 1: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+1}^r) \quad \text{if } n - rk \ll n^{\frac{r-1}{r}}; \quad (4)$$

$$\text{fixed } l: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } k \gg n^{\frac{r}{r-1}} \log^{\frac{1}{r-1}} n, \quad l \text{ is fixed.} \quad (5)$$

We do not express $\chi(KG_{n,k+l}^r)$ in terms of $\chi(KG_{n,k}^r)$, since the formulas are much uglier in the hypergraph case. Note that for $r = 2$ the bound (4) coincides with (1), while (5) improves on (2).

In this paper we prove the following bounds.

Theorem 1. *Let $p > 0$, $r \geq 2$, $t \in \mathbb{N}$, $\epsilon > 0$ be fixed. Then the following holds a.a.s.:*

$$l = 1, r = 2^t: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+1}^r) \quad \text{if } n - rk \ll n^{r/(r+1)} \log^{-1/(r+1)} n; \quad (6)$$

$$l = 1, r \neq 2^t: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+1}^r) \quad \text{if } n - rk \ll n^{\frac{r-1}{r}}; \quad (7)$$

$$\text{fixed } l, r = 2: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } k \gg (n \log n)^{1/l}, \quad l \text{ is fixed}; \quad (8)$$

$$\text{fixed } k, r = 2: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } l \gg (n \log n)^{1/k}, \quad k \text{ is fixed}; \quad (9)$$

$$\text{fixed } l, r = 3: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } k \gg \log^{1/(3l-4)} n, \quad l \text{ is fixed}; \quad (10)$$

$$\text{fixed } k, r = 3: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } l \gg \log^{2/(6k-11)} n, \quad k \text{ is fixed}; \quad (11)$$

$$\text{fixed } l, r > 3: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } k \gg \log^{\frac{1}{r(l-2)-1}} n, \quad l \text{ is fixed}; \quad (12)$$

$$\text{fixed } k, r > 3: \quad \chi(KG_{n,k}^r)(p) \geq \chi(KG_{n,k+l}^r) \quad \text{if } l \gg \log^{\frac{1}{r(k-1)-\frac{2r-1}{r-1}}} n, \quad k \text{ is fixed}; \quad (13)$$

We remark that the bounds stated in the theorem hold for Schrijver graphs and, more generally, for r -stable r -uniform Kneser hypergraphs, when $r = 2^t$ for some $t \in \mathbb{N}$.

In the graph case, we see that (6), (8), and (9) improve (1), (2) and (3), respectively. The inequality (7) coincides with the inequality (4).

The most interesting ones are, however, (10)–(13). They are much stronger than (5) and guarantee that the chromatic number $\chi(KG_{n,k}^r)$ drops no more than by a small additive term for polylogarithmic k .

One question that arises in this context is what makes the case $r = 2$ so different from the case $r > 2$? Can one obtain a bound similar to (10)–(13) for the case $r = 2$?

In the next section we present the general approach to the problem, and obtain the inequalities (6), (7). The approach, which is more adapted to our particular problem, is presented in Section 4. The rest of the inequalities from Theorem 1 are obtained there. In Section 6 we discuss some directions for further research.

3 Basic approach

In this section we discuss the general method, proposed to us by N. Alon, along with some of its corollaries to the case of Kneser and Schrijver graphs and hypergraphs. We prove the bounds (6) and (7) in this section.

3.1 Coloring random subgraphs of blow-ups of hypergraphs

We start with the following abstract theorem on hypergraph colorings, preceded by the definition of a class of hypergraphs in question. For an r -uniform hypergraph $H = (V, E)$ and a positive integer number m consider the m -blow-up $H(m)$ of H : $H(m) = (V', E')$, where $V' := V \times [m]$, and $E' := \{\{v_1 \times i_1, \dots, v_r \times i_r\} : \{v_1, \dots, v_r\} \in E, i_1, \dots, i_r \in [m]\}$. Informally speaking, we replace each vertex of the original hypergraph with an m -tuple, and each edge with an r -partite hypergraph with m vertices in each part.

We denote by $\mathcal{A}(H, m)$ the class of hypergraphs that can be obtained from $H(m)$ by identifying some vertices that do not belong to the same edge and do not arise from the same vertex of H . Formally, consider the class \mathcal{F} of functions $f : V' \rightarrow [n]$ for some n , such that:

1. For any $v \in V$ and $i \neq j$ we have $f(v \times i) \neq f(v \times j)$.
2. For any $e \in E'$, $v_1, v_2 \in e$ we have $f(v_1) \neq f(v_2)$.
3. The function f is surjective.

Then the class of hypergraphs $\mathcal{A}(H, m)$ is defined as follows:

$$\mathcal{A}(H, m) := \left\{ (f(V'), E_f) : f \in \mathcal{F}, E_f := \{\{f(v_1), \dots, f(v_r)\} : \{v_1, \dots, v_r\} \in E'\} \right\}.$$

We denote by $K^r(m, \dots, m)$ a complete r -partite r -uniform hypergraph with parts of size m . For any $0 < p < 1$ and a hypergraph G we define the random hypergraph $G(p)$, which has the same set of vertices and in which each edge from G is taken independently with probability p .

Theorem 2. *Let H be an r -uniform hypergraph with $\chi(H) = d + 1$. Fix a number $m \in \mathbb{N}$ and consider a hypergraph $G \in \mathcal{A}(H, m)$. Then for any coloring of G in d colors there is a subhypergraph $K^r(\lceil \frac{m}{d} \rceil, \dots, \lceil \frac{m}{d} \rceil) \subset G$ with all vertices colored in the same color.*

Moreover, for any $0 < p < 1$ we have

$$\Pr[\chi(G(p)) \leq d] \leq |E| \binom{m}{\lceil m/d \rceil}^r (1 - p)^{\lceil m/d \rceil^r}. \quad (14)$$

Proof. Consider any coloring of G in d colors. We construct a certain coloring of H based on the coloring of G . For each vertex $v \in H$ take its blow-up v_1, \dots, v_m in G and color v in the most popular color among v_1, \dots, v_m . It is clear that at least $\lceil \frac{m}{d} \rceil$ of v_i 's are colored in this color.

Since $\chi(H) > d$, we find a monochromatic edge of color κ in this coloring. In G this edge corresponds to an r -uniform r -partite subhypergraph $K^r(m, \dots, m)$. Choosing out of each part the vertices colored in color κ , we get the desired subhypergraph.

In view of the argument above, the event $[\chi(G(p)) \leq d]$ may occur only if one of the subhypergraphs $K^r(\lceil \frac{m}{d} \rceil, \dots, \lceil \frac{m}{d} \rceil)$ of the type described above is empty in the random hypergraph $G(p)$. The number of such subhypergraphs is bounded from above by $|E| \binom{m}{\lceil m/d \rceil}^r$, while the probability for each to be empty in $G(p)$ is $(1-p)^{\lceil m/d \rceil^r}$. Thus, we get the inequality (14) applying the union bound. \square

3.2 Numerical Corollaries for Kneser hypergraphs

For $n \geq (k+l)r$ the hypergraph $KG_{n,k}^r$ belongs to the family $\mathcal{A}(KG_{k+l}^r, \binom{k+l}{k})$. Indeed, correspond to each vertex $S \in \binom{[n]}{k+l}$ of $KG_{n,k+l}^r$ the family of subsets $\binom{S}{k}$. Put

$$t := \left\lceil \binom{k+l}{k} / d \right\rceil. \quad (15)$$

The following lemma gives the first (but not the strongest) general bound on the chromatic number of random Kneser hypergraphs.

Lemma 1. *For $n \geq (k+l)r$ we have $\chi(KG_{n,k}^r(p)) \geq d+1 := \chi(KG_{n,k+l}^r)$ a.a.s. if*

$$3r \left((k+l) \log \frac{n}{k} + t \log d \right) - pt^r \rightarrow -\infty \quad (16)$$

Proof. Remark that the number of edges in $KG_{n,k+l}^r$ is at most

$$\binom{n}{k+l}^r \leq \left(\frac{ne}{k} \right)^{(k+l)r}.$$

Therefore, applying the bound (14), we get that

$$\begin{aligned} \Pr[\chi(G(p)) \leq d] &\leq |E(KG_{n,k+l}^r)| \left(\binom{k+l}{k} / t \right)^r (1-p)^{t^r} \leq \left(\frac{ne}{k} \right)^{(k+l)r} (ed)^{rt} e^{-pt^r} \leq \\ &\exp \left[(k+l)r \left(1 + \log \frac{n}{k} \right) + rt(1 + \log d) - pt^r \right] \leq \exp \left[3r \left((k+l) \log \frac{n}{k} + t \log d \right) - pt^r \right]. \end{aligned}$$

The last expression tends to 0 by (16), which concludes the proof of the lemma. \square

What does Lemma 1 give for the cases discussed in Section 2? Doing some routine calculations (see the proofs of the corollaries in [21] for more details) we get that for fixed p, r and l the following holds a.a.s.:

$$l = 1: \quad \chi(KG_{n,k}^r(p)) \geq \chi(KG_{n,k+1}^r) \quad \text{if } n - rk \ll n^{\frac{r-1}{r}}; \quad (17)$$

$$\text{fixed } l: \quad \chi(KG_{n,k}^r(p)) \geq \chi(KG_{n,k+l}^r) \quad \text{if } k \gg n^{\frac{r}{lr-1}} \log^{\frac{1}{lr-1}} \quad \text{and } l \text{ is fixed.} \quad (18)$$

The first bound is the same as (1), (4) and (7), while the second one is the same as the bound (5). However, (18) is still a long way from the latter bounds in Theorem 2.

The trick in the case $l = 1, r = 2^t$ is to pass to r -stable Kneser hypergraphs. As in the case of (complete) Kneser hypergraphs, $KG_{n,k}^{r, r\text{-stable}} \in \mathcal{A}(KG_{n,k+l}^{r, r\text{-stable}}, \binom{k+l}{k})$, and, if $n - rk = o(n)$, it has fewer vertices and edges than $KG_{n,k}^r$. But how much fewer?

Proposition 1. *The number of vertices in the hypergraph $KG_{n,k}^{r, r\text{-stable}}$ is at most $\binom{n-(r-1)k+1}{k}$.*

Proof. The vertices of Schrijver graph are the k -subsets $\{i_1, \dots, i_k\}$ of $[n]$ that satisfy $i_{j+1} - i_j \geq s$ for each $j = 0, \dots, k-1$, as well as $i_1 + n - i_k \geq s$, provided that $1 \leq i_1 < \dots < i_k \leq n$. Let us count the number $f(n, k)$ of k -sets satisfying all these restrictions except $i_1 + n - i_k \geq s$. This number will clearly be an upper bound on $|V(KG_{n,k}^{r, r\text{-stable}})|$.

It is easy to see that this quantity satisfies the following recursive formula: $f(n, k) = f(n-1, k) + f(n-r, k-1)$, as well as the condition $f(rk-1, k) = 1$. The function $\binom{n-(r-1)k+1}{k}$ satisfies both the recursive formula and the initial condition. \square

We aim to prove that $\chi(KG_{n,k}^{r, r\text{-stable}}(p)) \geq d+1 := \chi(KG_{n,k+1}^{r, r\text{-stable}})$ for the widest possible range of parameters. From Proposition 1 we get that the number of edges in $KG_{n,k+1}^{r, r\text{-stable}}$ is at most

$$\binom{n-(r-1)k+1}{k}^r = \binom{k+O(d)}{k}^r = \left(\frac{O(n)}{O(d)}\right)^r = e^{O(d \log \frac{n}{d})}.$$

Thus, instead of (16) it is sufficient to show that (recall that p and r are fixed)

$$d \log \frac{n}{d} + t \log d \ll pt^r. \quad (19)$$

We have $t = \Theta(\frac{n}{d})$ and $d = \Theta(n - rk)$. Doing some routine calculations again, we get that

$$\mathbf{l = 1:} \quad \chi(KG_{n,k}^{r, r\text{-stable}})(p) \geq \chi(KG_{n,k+1}^{r, r\text{-stable}}) \quad \text{if } n - rk \ll n^{r/(r+1)} \log^{-1/(r+1)} n.$$

Since for $r = 2^t$, where $t \in \mathbb{N}$, we have $\chi(KG_{n,k+1}^{r, r\text{-stable}}) = \chi(KG_{n,k+1}^r)$, we get (6).

4 The approach refined

The crucial step in the proof of Theorem 2 is to get a monochromatic edge of $KG_{n,k+l}^r$, related to the coloring of $KG_{n,k}^r$. The main limitation of the method from the previous section is related to this step. We have to assume that (in the worst case) among the vertices of the m -blow-up of the monochromatic edge all colors are represented in approximately the same proportion. This is why we can only guarantee the majority color class

to have the size at least $\frac{m}{d}$. On the other hand, to get a good bound on the probability, we need to work with color classes of growing size. Therefore, the approach from the previous section is bound to work only for $m \gg d$, or, in terms of Kneser hypergraphs, for $\binom{k+l}{k} \gg \chi(KG_{n,k+l}^r)$.

In this section we are going to partially overcome the aforementioned difficulty. We assume that $n \gg k+l$ and that $r \geq 2$ is fixed for the rest of the section. Assume that we want to prove that $\chi(KG_{n,k}^r(p)) \geq d+1 := \chi(KG_{n,k+l}^r)$ a.a.s. Fix a coloring κ of $\binom{[n]}{k}$ in d colors. For each subset $S \subset [n]$ of size at least k , define the color of S to be the most popular color among its subsets. We have thus defined the coloring κ' of $KG_{n,k'}^r$ for all $k' \geq k$. Put $u = \lfloor \log_2 \frac{n}{k+l^\beta} \rfloor$ and consider the numbers

$$q_0 = k+l, \quad q_i := 2^i(k+l^\beta), \quad \text{where } i = 1, \dots, u, \quad \text{and } \beta = \begin{cases} 1, & r = 2 \\ \frac{r}{r-1}, & r \geq 3. \end{cases} \quad (20)$$

Thus, by definition, no q_i is bigger than n . The numbers q_i will play the role of the sizes of subsets that form vertex sets of Kneser hypergraphs. We will use the bound $q_{i+1} \leq 2l^{\alpha-1}q_i$. Next, for each $i = 0, \dots, u$, define the following two numbers:

$$t_i := \left\lceil \frac{\binom{q_i}{k}}{d} \right\rceil, \quad z_i := \left\lceil \frac{\binom{q_i}{k}}{2sq_i} \right\rceil, \quad \text{where } s = \begin{cases} (2r+1)l^{\beta-1}, & \text{if } r = 2, 3; \\ (2r+1)l^{\beta-1}k, & \text{if } r > 3. \end{cases} \quad (21)$$

As one can see, t_0 is equal to t from the previous section. Both t_i and z_i will play roles of the sizes of popular colors among the k -subsets of a certain q_i -element set.

The following lemma is central for this subsection.

Lemma 2. *For any coloring κ of $KG_{n,k}^r$ in d colors there is a color α for which one of the following holds.*

1. *There exists $i \in \{0, \dots, u\}$ and r pairwise disjoint subsets $A_1, \dots, A_r \in \binom{[n]}{q_i}$, such that at least z_i subsets from each $\binom{A_j}{k}$, $j = 1, \dots, r$, are colored in α .*
2. *There exists $i \in \{0, \dots, u-1\}$ and r pairwise disjoint subsets $A_1 \in \binom{[n]}{q_i}$, $A_2, \dots, A_r \in \binom{[n]}{q_{i+1}}$, such that at least t_i subsets from $\binom{A_1}{k}$ and z_{i+1} subsets from each $\binom{A_j}{k}$, $j = 2, \dots, r$, are colored in α .*

Proof. We start with analyzing the colorings of $KG_{n,k}^r$ in d colors. For each $i = 0, \dots, u$ with each set $A \in \binom{[n]}{q_i}$ associate the set X_A of the colors that are used at least $\frac{t_i}{2}$ times on the family $\binom{A}{k}$ (recall the definitions (20), (21)). Note that at least half of the vertices of $\binom{A}{k}$ is colored by colors from X_A . We have two possibilities for a given i : either for each set $A \in \binom{[n]}{q_i}$ there is a color that is used for z_i vertices on $\binom{A}{k}$, or there is a set $A \in \binom{[n]}{q_i}$ such that $|X_A| > sq_i$. This is obvious in case $z_i \leq t_i$. If $z_i > t_i$ (which is typically the case), then the negation of both statements implies that in X_A there are less than sq_i

colors, each of cardinality strictly less than z_i . But X_A account for the coloring of at least half the vertices in $\binom{A}{k}$, which is $sq_i z_i$. This is a contradiction.

If for each q_i -element set A there is a color in X_A used at least z_i times, then, arguing as in the proof of Theorem 2, we get the first possibility from the lemma.

If not, then fix *the largest* index i , for which there exist a q_i -element set A with $|X_A| > sq_i$, and choose such A . Clearly, $i \leq u - 1$, otherwise we have more than d colors in X_A . Put $Y := [n] \setminus A$ and consider the majority coloring κ' of the q_{i+1} -element subsets of Y . The Kneser hypergraph, induced on these sets, we denote by $KG_{Y, q_{i+1}}^r$.

We claim that at least one of the two holds: either the coloring of $KG_{Y, q_{i+1}}^r$ is not proper, or in the coloring κ' of $KG_{Y, q_{i+1}}^r$ there is a color from X_A used for an $(r-1)$ -tuple of pairwise disjoint q_{i+1} -element sets. If the first possibility takes place, we again get the first possibility from the lemma (remark that for any q_{i+1} -set B the set X_B has a color class of size z_{i+1} by the definition of i). If the second possibility takes place, then we arrive at the second possibility from the lemma. Thus, we are left to show that one of the two statements claimed in the beginning of the paragraph is true.

Recall that $q_{i+1} \leq 2lq_i$. Assume that neither of the two options takes place. In the case $r = 2$, this simply means that $KG_{Y, q_{i+1}}$ is properly colored in less than $d - sq_i$ colors (the colors from X_A are not used in the coloring). Therefore, $n - 2k - 2l + 1 - sq_i = d - sq_i > \chi(KG_{Y, q_{i+1}}) = (n - q_i) - 2q_{i+1} + 2$, which is equivalent to $(s - 5)q_i + 2(k + l) + 1 < 0$. But, by the definition (21), $s = 5$ when $r = 2$. We arrive at a contradiction.

In the case $r \geq 3$ this means that $KG_{Y, q_{i+1}}^r$ is properly colored, and that for each color $\alpha \in X_A$ the collection of sets colored in α contains at most $r - 2$ pairwise disjoint sets.

If $r = 3$, then construct a new coloring of $KG_{Y, q_{i+1}}^3$, grouping the colors from X_A into pairs forming a single new color. Remark that the new coloring uses at most $d - \frac{sq_i}{2}$ colors, and that this coloring is still proper, since none of the newly formed colors has more than two pairwise disjoint sets. Therefore, $\chi(KG_{n, k+l}^3) - 1 - \frac{sq_i}{2} \geq \chi(KG_{Y, q_{i+1}}^3)$.

In general, we have

$$\chi(KG_{n, k+l}^r) - \chi(KG_{Y, q_{i+1}}^r) = \left\lceil \frac{n - (k + l - 1)r}{r - 1} \right\rceil - \left\lceil \frac{n - q_i - (q_{i+1} - 1)r}{r - 1} \right\rceil < \frac{rq_{i+1} + q_i}{r - 1} = \frac{(2r + 1)l^{\beta-1}q_i}{r - 1}. \quad (22)$$

Thus, for $r = 3$ we conclude that $\frac{sq_i}{2} < \frac{(2r+1)lq_i}{2}$, which contradicts (21).

Finally, consider the case $r > 3$. We again construct a new coloring of $KG_{Y, q_{i+1}}^r$ that uses relatively few colors, via the following procedure. We split the colors from X_A into groups of size $k(r - 2) + 1$. In each group we choose one color α and split vertices (sets) from $KG_{Y, q_{i+1}}$ of color α into $k(r - 2)$ groups of pairwise intersecting sets. The existence of such a splitting is easy to verify. Take a maximum family of pairwise disjoint sets in $KG_{Y, q_{i+1}}$ colored in α . It has size most $r - 2$, and thus it covers a set $U \subset Y$ of cardinality at most $(r - 2)k$. Each other set of color α in Y must intersect U . We then simply split

all sets of color α into families $\mathcal{K}_j, j = 1, \dots, |U|$ depending on which element from U it contains.

Next, we adjoin each of the families \mathcal{K}_j to one of the remaining $k(r-2)$ colors in the group. At the end we get a proper coloring, since none of the newly formed colors contain more than $r-1$ pairwise disjoint sets. The number of colors used in the new coloring is less than $d - \lfloor \frac{sq_i}{k(r-2)+1} \rfloor \leq d - \lfloor \frac{sq_i}{k(r-1)} \rfloor$. Thus, comparing the inequality $\chi(KG_{n,k+l}^r) - 1 - \lfloor \frac{sq_i}{k(r-1)} \rfloor \geq \chi(KG_{Y,q_{i+1}}^3)$ with the inequality (22), we get

$$\left\lfloor \frac{s}{k(r-1)} \right\rfloor + 1 < \frac{(2r+1)l^{\beta-1}q_i}{r-1},$$

which contradicts the definition (21). \square

Lemma 2 tells us that if there is a proper d -coloring of $KG_{n,k}^r(p)$, then there are subsets A_1, \dots, A_r as in the lemma, such that the induced subgraph of $KG_{n,k}^r(p)$ on these subsets has no edge. In what follows we calculate the probabilities of these events.

Assume that the first possibility from Lemma 2 for a certain i takes place. The probability of the corresponding event for the random hypergraph is at most

$$\binom{n}{q_i}^r \binom{\binom{q_i}{k}}{z_i}^r (1-p)^{z_i^r}. \quad (23)$$

If the second possibility from Lemma 2 for a certain i takes place, then the probability of the corresponding event for the random hypergraph is at most

$$\binom{n}{q_i} \binom{n}{q_{i+1}}^{r-1} \binom{\binom{q_i}{k}}{t_i} \binom{\binom{q_{i+1}}{k}}{z_{i+1}}^{r-1} (1-p)^{t_i z_{i+1}^{r-1}} \leq \binom{n}{q_{i+1}}^r \binom{\binom{q_{i+1}}{k}}{z_{i+1}}^r (1-p)^{t_i z_{i+1}^{r-1}}. \quad (24)$$

Typically, the last expression in (24) is much bigger than that in (23). Note that the total number of events is $2u+1 \leq 2\log n$. Therefore, if each has probability less than $\frac{1}{n}$, say, then we have $\chi(KG_{n,k}^r(p)) \geq d+1$ a.a.s. We remark that this condition on the probability of a single event is by no means restrictive for as, as we are manipulating with expressions of much higher order of growth. The following analogue of Lemma 1, giving the general bound on the chromatic number of random Kneser hypergraphs, is proven by the same calculations:

Lemma 3. *For $n \geq (k+l)r$ we have $\chi(KG_{n,k}^r(p)) \geq d+1 := \chi(KG_{n,k+l}^r)$ a.a.s. if for each $i = 0, \dots, u$ we have*

$$3r(q_i \log n + z_i \log(2sq_i)) - pz_i^r \rightarrow -\infty \quad (25)$$

$$3r(q_{i+1} \log n + z_{i+1} \log(2sq_{i+1})) - pt_i z_{i+1}^{r-1} \rightarrow -\infty \quad (26)$$

In what follows, we assume that $p > 0$ is fixed, and that $k, l \geq 2$. We have $\log q_i = \Omega(\log(2sq_{i+1}))$ for any $i \geq 0$. Then the inequalities (25) and (26) follow from

$$z_i^{r-1} \gg \log q_i, \quad z_i^r \gg q_i \log n, \quad t_i z_{i+1}^{r-2} \gg \log q_i, \quad t_i z_{i+1}^{r-1} \gg q_{i+1} \log n. \quad (27)$$

We have $z_{i+m}/z_i = \Omega(q_{i+m}/q_i)$ for any $r \geq 2, k \geq 2, m = 1, \dots, u-i$. Therefore, for $r \geq 3$ it is sufficient to verify the inequalities (27) for $i = 0$. For $r = 2$ it is sufficient to verify the inequalities 1, 2, and 4 from (27) for $i = 0$, and the inequality $t_0 \geq \log n$.

For $r = 2, 3$ we have $s = O(l^{1/2})$, which results in $z_i = O\left(\frac{\binom{q_i}{k}}{l^{1/2}q_i}\right) \gg \log q_i$. Therefore, (27) for $r = 2, 3$ reduces to the following:

$$z_0^r \gg (k+l) \log n, \quad t_i z_i^{r-2} \gg \log q_i, \quad t_0 z_1^{r-1} \gg (k+l) \log n, \quad (28)$$

where for $r \geq 3$ one has to verify the second inequality only for $i = 0$. Let $r = 2$. Looking at the definitions (21), it is clear that the second condition is the most restrictive. The following inequality is sufficient to satisfy (28) and implies both (8) and (9):

$$\binom{k+l}{k} \gg n \log n. \quad (29)$$

For $r = 3$, replacing the t_0 factor with 1, we conclude that the second inequality in (28) always holds and that (28) follows from

$$\binom{k+l}{k}^3 \gg (k+l)^4 l^{3/2} \log n, \quad \left(2(k+l^{3/2})\right)^2 \gg (k+l)^3 l \log n. \quad (30)$$

For fixed l the first condition is clearly more restrictive, and we get that (30) holds for $k \gg \log^{1/(3l-4)} n$. For fixed k the first inequality is more restrictive again, and we get that it holds for $l \gg \log^{1/(3k-9/2)} n$. This gives the inequalities (10) and (11).

For $r > 3$ we have to assume $l \geq 3$ in order to get any good lower bound on z_i : for $l = 2$ we have $z_0 = 1$. But for $l \geq 3$ we again have $z_i \gg \log q_i$, so it is again enough to verify (28). Replacing t_0 with 1, we get that (28) is implied by

$$\binom{k+l}{k}^r \gg k^r (k+l)^{r+1} l^{r/(r-1)} \log n, \quad \left(2(k+l^{r/(r-1)})\right)^{r-1} \gg k^{r-1} (k+l)^r l \log n. \quad (31)$$

Similarly to the case $r = 3$, for both fixed l and fixed k the first condition is more restrictive. For fixed l we get that (31) holds for $k \gg \log^{\frac{1}{r(l-2)-1}} n$. For fixed k it holds for $l \gg \log^{\frac{1}{r(k-1)-\frac{2r-1}{r-1}}} n$. This gives the inequalities (12) and (13).

5 Simple lower bounds

In this section we present simple upper bounds for $\chi(KG_{n,k}^r(p))$ and compare them with the results of Theorem 1. If there exist a set $A \subset [n]$ of size $rk + l$, such that $KG^r(n, k)(p)|_A$ is an empty graph, then, coloring A into color 0 and the rest as in the standard coloring of $KG^r(n, k)$, we get that $\chi(KG_{n,k}^r(p)) \leq \chi(KG_{n,k}^r) - \lfloor l/(r-1) \rfloor$. To estimate the probability of having such A , we find n sets of size $l + 2k$ in $[n]$, which have pairwise intersections of size at most 1, and, and calculate the probability that one of those becomes empty. Note that the events for different sets are independent. The probability is

$$\left(1 - (1-p)^{\prod_{i=1}^r \binom{l+ik}{k}}\right)^n \leq e^{-n(1-p)^{\prod_{i=1}^r \binom{l+ik}{k}}}.$$

Therefore, if

$$(1-p)^{\prod_{i=1}^r \binom{l+ik}{k}} n \rightarrow \infty, \quad (32)$$

then a.a.s. there exists such a set. If p, r, k are fixed, then this condition is satisfied if for sufficiently large constant we have $e^{cl^{rk}} = o(n)$, which implies that we can take $l = \Omega(\log^{\frac{1}{rk}} n)$. This shows that bounds (11), (13) are essentially tight: the difference between the lower and the upper bounds are in the degree of the logarithm.

If p, r , and l are fixed, then the situation is more interesting. The condition (32) is satisfied if $e^{c^{r^2k}} = o(n)$, which could be fulfilled for $k = \Omega(\log \log n)$. This is very different from the bounds (10), (12). Of course, in the graph case ($r = 2$) the gap between the upper and lower bounds is even bigger.

6 Discussion

In [21] Kupavskii asked whether it is true that for some $k = k(n)$ a.a.s. we have $\chi(KG_{n,k}(1/2)) = \chi(KG_{n,k})$. This question remains wide open for all meaningful values of k (by that we mean that $n - 2k \rightarrow \infty$), with current methods not allowing to attack it. We also ask a similar question for Kneser hypergraphs. This may be easier to show in the hypergraph case. Indeed, when the bound (7) is applicable, then for sufficiently large r and most n the difference between the chromatic number of $KG_{n,k}^r$ is guaranteed to be at most 1.

The huge difference in the bounds between the cases $r = 2$ and $r \geq 3$ demands some exploration. What is the correct order of growth of k needed to guarantee that the chromatic number of a Kneser graph a.a.s. drops by an additive term only when passing to a random subgraph? We conjecture that the following should be true:

Conjecture. *For any fixed $p > 0$ we have $\chi(KG_{n,k}(p)) \geq \chi(KG_{n,k}) - 4$ for $k \gg \log n$.*

So far most of the research in this direction was concerned with lower bounds. But what about upper bounds, or, stated in a more convenient way, lower bounds for the

expression $\chi(KG_{n,k}^r) - \chi(KG_{n,k}^r(p))$? In the previous section we showed that for fixed k and for $r \geq 3$ we can obtain lower bounds similar to the upper bounds given by (11), (13). The case fixed k and $r = 2$ seems to be troubling again, as the lower bounds for $\chi(KG_{n,k}^r) - \chi(KG_{n,k}^r(p))$ that are in sight are logarithmic, while the upper ones, provided by (9), are polynomial. We also have a huge gap between the upper and the lower bounds on k , for which the difference between $\chi(KG_{n,k}^r)$ and $\chi(KG_{n,k}^r(p))$ is at most a fixed constant l , as we show in the previous section.

Finally, we remark that the method from Theorem 2 may be applied to the following class of Kneser hypergraphs of arbitrary set systems. Assume that $\mathcal{F} \subset \binom{[n]}{k+l}$ is an arbitrary family of $k+l$ -element sets. We form a family \mathcal{H} of all k -element subsets, contained in at least one set from \mathcal{F} (this is the so-called k -th shadow of \mathcal{F}). Denote by $KG^r(\mathcal{H})$ the r -uniform Kneser hypergraph on \mathcal{H} . Then the equation (14) tells us that $\chi(KG^r(\mathcal{H})(p)) \geq d+1 := \chi(KG^r(\mathcal{F}))$ with probability at least

$$1 - |\mathcal{H}| \left(\binom{k+l}{k} \right)^r (1-p)^{\left\lceil \frac{k+l}{d} \right\rceil^r}.$$

The following question seems worth to explore: are there any interesting classes of graphs or hypergraphs, for which the topological bounds (as the ones proven in [21] and [1]) work, while the present combinatorial approach fails?

References

- [1] M. Alishahi, H. Hajiabolhassan, *Chromatic Number of Random Kneser Hypergraphs*, arXiv:1607.07432v1
- [2] N. Alon, L. Drewnowski, T. Łuczak, *Stable Kneser hypergraphs and ideals in \mathbb{N} with the Nikodým property*, Proc. Amer. Math. Soc. 137 (2009), N2, 467–471.
- [3] N. Alon, J. Spencer, *The probabilistic method*, Wiley–Interscience Series in Discrete Mathematics and Optimization, Second Edition, 2000.
- [4] N. Alon, P. Frankl, L. Lovász, *The chromatic number of Kneser hypergraphs*, Transactions of the American Mathematical Society, 298 N1 (1986), 359–370.
- [5] I. Bárány, *A short proof of Kneser’s conjecture*, Journal of Combinatorial Theory, Ser. A 25 (1978), N3, 325–326.
- [6] B. Bollobás, *Random Graphs*, Cambridge University Press, Second Edition, 2001.
- [7] J. Balogh, B. Bollobás, B.P. Narayanan, *Transference for the Erdős-Ko-Rado theorem*, preprint

- [8] L.I. Bogolyubskiy, A.S. Gusev, M.M. Pyaderkin, A.M. Raigorodskii, *The independence numbers and the chromatic numbers of random subgraphs of some distance graphs*, Mat. Sb., To appear.
- [9] L.I. Bogolyubskiy, A.S. Gusev, M.M. Pyaderkin, A.M. Raigorodskii, *Independence numbers and chromatic numbers of random subgraphs in some sequences of graphs*, Doklady Math. 457 (2014), 383–387.
- [10] B. Bollobás, B.P. Narayanan, A.M. Raigorodskii, *On the stability of the Erdős-Ko-Rado theorem*, arXiv:1408.1288.
- [11] V.L. Dol’nikov, *Transversals of families of sets*, in Studies in the theory of functions of several real variables (Russian), Yaroslav. Gos. Univ., Yaroslavl’, 109 (1981), 30–36.
- [12] P. Erdős, *A problem on independent r -tuples*, Ann. Univ. Sci. Budapest. 8 (1965) 93–95.
- [13] P. Erdős, *Problems and results in combinatorial analysis*, Colloq. Internat. Theor. Combin. Rome (1973), Acad. Naz. Lincei, Rome (1976), 3–17.
- [14] P. Erdős, C. Ko, R. Rado, *Intersection theorems for systems of finite sets*, The Quarterly Journal of Mathematics, 12 (1961) N1, 313–320.
- [15] P. Frankl, *Improved bounds for Erdős’ Matching Conjecture*, Journ. of Comb. Theory Ser. A 120 (2013), 1068–1072.
- [16] P. Frankl, A. Kupavskii, *Two problems of P. Erdős on matchings in set families*, arXiv:1607.06126v1
- [17] P. Frankl, A. Kupavskii, *Families with no s pairwise disjoint sets*, arXiv:1607.06122v1
- [18] M. Kneser, *Aufgabe 360*, Jahresbericht der Deutschen Mathematiker-Vereinigung 2 (1955), 27.
- [19] I. Kříž, *Equivariant cohomology and lower bounds for chromatic numbers*, Trans. Amer. Math. Soc. 333 (1992), 567–577
- [20] I. Kříž, *A correction to “Equivariant cohomology and lower bounds for chromatic numbers” (Trans. Amer. Math. Soc. 333 (1992), 567–577)*, Trans. Amer. Math. Soc., 352 (2000), N4, 1951–1952.
- [21] A. Kupavskii, *On random subgraphs of Kneser and Schrijver graphs*, Journal of Combinatorial Theory, Series A 141 (2016), 8–15. arXiv:1502.00699v1
- [22] L. Lovasz, *Kneser’s conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A, 25 (1978), N 3, 319–324.
- [23] J. Matoušek, *Using the Borsuk-Ulam theorem*, Springer, 2003.
- [24] J. Matoušek, *On the chromatic number of Kneser hypergraphs*, Proc. Amer. Math. Soc. 130 (2002), N9, 2509–2514.

- [25] F. Meunier, *The chromatic number of almost stable Kneser hypergraphs*, J. Combin. Theory Ser. A 118 (2011), N6, 1820–1828.
- [26] A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wiskd., III. Ser., 26 (1978), 454–461.
- [27] G. M. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. 147 (2002), 671–691. Erratum *ibid.*, 163 (2006), 227–228.